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On the Structure of Forms, and the Algebraical Theory of n-Lines.

By O. E. Glenn.

Factorable ternary quantics, representing n-lines, have an invariant theory which is in some respects analogous to binary invariant theory. The theory of degenerate forms lends considerable aid also to the study of non-factorable forms. In this paper we propose to develop the theory of factorable forms considerably farther than was done in a paper on the subject published by the present writer in Vol. XXXII, No. 1, of this Journal,* and to make a series of applications of this theory. Sections 1, 2, 3, 4 are devoted to ternary factor theorems. In § 5 an introduction to an invariant theory of n-lines from the standpoint of matrices is given. Resultant and discriminant matrices of forms representing n-lines are constructed. Section 6 contains a theory of rational partial fractions from the point of view of the Aronhold operator.

§ 1. *Multiple Linear Factors.*

The number of terms in an elementary symmetric function of m groups of any p homogeneous variables is equal to the number of distinct permutations of the variables occurring in any one term, when the subscripts are removed. Thus if the groups are

$$\begin{array}{l} P_1(q_1, r_1, s_1), \\ P_2(q_2, r_2, s_2), \\ \cdots\cdots\cdots\cdots, \\ P_m(q_m, r_m, s_m), \end{array}$$

we are led by a simple proof to the relation

$$\Sigma \frac{\partial}{\partial r_1}(\Sigma q_1 q_2 \cdots q_r r_{r+1} r_{r+2} \cdots r_{m-k} s_{m-k+1} \cdots s_m) s_1 \\ = (k+1) \Sigma q_1 q_2 \cdots q_r r_{r+1} r_{r+2} \cdots r_{m-k-1} s_{m-k} \cdots s_m. \quad (1)$$

* "The Theory of Degenerate Algebraical Curves and Surfaces." The paper contains a short bibliography of the subject, to which may be added, Brioschi: "Sulla Condizioni, etc.," *Annali di Mathematica*, Ser. 12, Vol. VII; Thaer: "Ueber die Zerlegbarkeit einer ebenen Linie, u. s. w.," *Math. Annalen*, Vol. XIV; Bes: "Décomposition de la forme ternaire du troisième degré," *Math. Annalen*, Vol. LIX (1904).

Hence, when the ternary form

$$f_{3m} = a_{0m}\, y^m + \sum_{r=0}^{m-1} (a_{m-rr}\, x^{m-r} + a_{m-r-1r}\, x^{m-r-1} + \ldots + a_{0r})\, y^r,$$

which we write under the non-homogeneous notation

$$f_{3m} = x^m \phi_{0\,y/x} + x^{m-1} \phi_{1\,y/x} + \ldots + x\,\phi_{m-1\,y/x} + \phi_m,$$

is decomposable into m linear factors, its coefficients are connected with the coefficients of these linear forms, as

$$\rho_i = q_i\, y + r_i\, x + s_i \qquad (i = 1, 2, \ldots, m),$$

by the following relations, among others:

$$\Sigma \frac{\partial a_{m-r-kr}}{\partial r_1} s_1 = (k+1)\, a_{m-r-k-1r} \begin{pmatrix} r = 0, 1, \ldots, m-1 \\ k = 0, 1, \ldots, m-1 \\ \pi q_i = a_{0m} = 1 \end{pmatrix}. \tag{2}$$

Assume that r_1 is a root of multiplicity α_1 of

$$\phi_{0-r} \equiv r^m - a_{1m-1} r^{m-1} + a_{2m-2} r^{m-2} - \ldots + (-1)^m a_{m0} = 0.$$

Then evidently r_1 is a root of multiplicity $\alpha_1 - \varkappa$ of

$$\phi_{\varkappa-r} = 0 \qquad (\varkappa = 0, 1, 2, \ldots, \overline{\alpha_1 - 1}).$$

Now, when the coefficients a of $\phi_{\varkappa-r}$ are expressed in terms of the variables r_i, s_i, we write

$$\phi_\varkappa = \phi_{\varkappa-r}(a) = I_\varkappa(r_i, s_i; r) = I_\varkappa,$$

and then $I_\varkappa(r_i, s_i; r_1) \equiv 0$. By replacing r_i, r in $I_\varkappa$ by $r_i + \lambda s_i$, $r + \lambda s$ respectively, expanding by Taylor's theorem and using (2), we get

$$\begin{aligned} &\sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \left(\Sigma \frac{\partial}{\partial r_1} s_1 + \frac{\partial}{\partial r} s \right)^h I_\varkappa \\ &= \sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \left(\Sigma \frac{\partial}{\partial a_{m-r-kr}} (k+1)\, a_{m-r-k-1r} + \frac{\partial}{\partial r} s \right)^h \phi_\varkappa \\ &= \sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \sum_{i=0}^{h} \binom{h}{i} s^{h-i} \frac{\partial^{h-i} \Delta_k^i \phi_\varkappa}{\partial r^{h-i}} \\ &= \sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \sum_{i=0}^{h} (-1)^i \binom{h}{i} (\varkappa+1)(\varkappa+2) \ldots (\varkappa+i)\, s^{h-i} \frac{\partial^{h-i} \phi_{\varkappa+i}}{\partial r^{h-i}}, \end{aligned} \tag{3}$$

where

$$\Delta_k = \sum_k \left(a_{m-k-10} \frac{\partial}{\partial a_{m-k0}} + a_{m-k-21} \frac{\partial}{\partial a_{m-k-1_1}} + \ldots + a_{0\,m-k-1} \frac{\partial}{\partial a_{1\,m-k-1}} \right) (k+1).$$

The first non-vanishing term of this series, when $r = r_1$, is the term where $h = \alpha_1 - \varkappa$. Placing $\varkappa = 0$, we have from this term the result that the roots s_i corresponding to the α_1 equal r_i are the α_1 roots of the equation

$$\frac{\partial^{\alpha_1}\phi_{0-r_1}}{\partial r_1^{\alpha_1}} s^{\alpha_1} - \alpha_1 \frac{\partial^{\alpha_1-1}\phi_{1-r_1}}{\partial r_1^{\alpha_1-1}} s^{\alpha_1-1} + \alpha_1(\alpha_1 - 1)\frac{\partial^{\alpha_1-2}\phi_{2-r_1}}{\partial r_1^{\alpha_1-2}} s^{\alpha_1-2} - \ldots\ldots + (-1)^{\alpha_1} \underline{|\alpha_1}\, \phi_{\alpha_1-r_1} = 0. \tag{4}$$

Similar results hold for a root r_i of ϕ_{0-r} of multiplicity $\alpha_i\,(i = 1, 2, \ldots, t)$. That is, we have for the homogeneous

$$f_{3m} = x_2^m \phi_{0\,x_1/x_2} + x_2^{m-1} x_3 \phi_{1\,x_1/x_2} + x_2^{m-2} x_3^2 \phi_{2\,x_1/x_2} + \ldots\ldots + \phi_{m\,x_1/x_2} x_3^m:$$

THEOREM 1. *The linearly factorable homogeneous ternary form f_{3m}, whose leading binary ϕ_{0-r} has r_i as a root of multiplicity α_i ($i = 1, 2, \ldots, t$; $\alpha_1 + \alpha_2 + \ldots + \alpha_t = m$), can be factored into factors of the respective orders $\alpha_1, \alpha_2, \ldots, \alpha_t$, which are rational and integral in the coefficients of the form f_{3m} itself on the one hand, and in the quantities r_i on the other, linear in the coefficients; according to the formula*

$$f_{3m} = \prod_{i=1}^{t}\left[\frac{\partial^{\alpha}\phi_{0-r_i}}{\partial r_i^{\alpha_i}} x^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1}\phi_{1-r_i}}{\partial r_i^{\alpha_i-1}} x^{\alpha_i-1} y_i + \alpha_i(\alpha_i - 1)\frac{\partial^{\alpha_i-2}\phi_{2-r_i}}{\partial r_i^{\alpha_i-2}} x^{\alpha_i-2} y_i^2 + \ldots\ldots + (-1)^{\alpha_i} \underline{|\alpha_i}\, \phi_{\alpha-r_i} y_i^{\alpha_i}\right] \quad (x_1 + r_i x_2/x_3 = x/y_i). \tag{5}$$

As an application of this theorem consider the problem of resolving the factorable ternary quartic f_{34}, where

$$x_2^4 \phi_{0\,x_1/x_2} = a_{400} x_1^4 + a_{310} x_1^3 x_2 + a_{220} x_1^2 x_2^2 + a_{130} x_1 x_2^3 + a_{040} x_2^4, \text{ etc.,}$$

and

$a_{400} = 1$, $a_{301} = a + 3$, $a_{202} = 3a - 4$, $a_{103} = -4a - 12$, $a_{004} = -12a$,
$a_{310} = 9$, $a_{211} = 7a + 19$, $a_{112} = 13a - 22$, $a_{013} = -14a - 24$,
$a_{220} = 30$, $a_{121} = 16a + 40$, $a_{022} = 14a - 28$,
$a_{130} = 44$, $a_{031} = 12a + 28$,
$a_{040} = 24$.

We find the roots of $\phi_{0\,x_1/x_2} = 0$ to be $+r_1 = +2$, $+r_2 = 3$; $\alpha_1 = 3$, $\alpha_2 = 1$. Equation (4) and the corresponding one for s_2, α_2 are then computed. They are

$$s^3 - (a+1)s^2 + (a-6)s + 6a = 0, \qquad s - 2 = 0.$$

Their roots are $s_1 = -2$, $s_2 = 3$, $s_3 = a$; $s_4 = 2$. Hence,

$$f_{34} = (x_1 + 2x_2 - 2x_3)(x_1 + 2x_2 + 3x_3)(x_1 + 2x_2 + a x_3)(x_1 + 3x_2 + 2x_3).$$

§ 2. *Extension of Method for Quaternary Forms.*

For elementary symmetric functions of m groups of four variables we have

$$\Sigma \frac{\partial}{\partial r_1}(\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{j-k-l+2} \cdots r_{m-j-k}\, s_{m-j-k+1} \cdots s_{m-j}\, t_{m-j+1} \cdots t_m)\, t_1$$
$$= (j+1)\,\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{m-j-k-l+2} \cdots r_{m-j-k-1}\, s_{m-j-k} \cdots s_{m-j-1}\, t_{m-j} \cdots t_m\,,$$

$$\Sigma \frac{\partial}{\partial r_1}(\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{m-j-k-l+2} \cdots r_{m-j-k}\, s_{m-j-k+1} \cdots s_{m-j}\, t_{m-j+1} \cdots t_m)\, s_1$$
$$= (k+1)\,\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{m-j-k-l+2} \cdots r_{m-j-k+1}\, s_{m-j-k} \cdots s_{m-j}\, t_{m-j+1} \cdots t_m\,,$$

and with

$$f_{4m} = \sum_{l=0}^{m} \Big[\sum_{k=0}^{m-l} (a_{l\,m-k-l\,k}\, x^l + a_{l-1\,m-k-l\,k}\, x^{l-1} + \cdots + a_{0\,m-k-l\,k})\, z^k\Big]\, y^{m-k-l}$$
$$= \prod_{i=1}^{m} (q_i y + r_i x + s_i z + t_i),$$

there results

$$\left.\begin{aligned} \Sigma \frac{\partial a_{l\,m-j-k-l\,k}}{\partial r_1}\, t_1 &= (j+1)\, a_{l-1\,m-j-k-l\,k}\,, \\ \Sigma \frac{\partial a_{l\,m-j-k-l\,k}}{\partial r_1}\, s_1 &= (k+1)\, a_{l-1\,m-j-k-l\,k+1} \end{aligned}\right\} \qquad (6)$$

$$(j = 0, 1, \ldots, m-1;\; k = 0, 1, \ldots, m-1;\; l = 1, 2, \ldots, m).$$

The terms of a quaternary form may be arranged, under a homogeneous notation, as follows:

$$f_{4m} = x_2^m \phi_{0\,x_1/x_2} + x_2^{m-1} x_3 \phi^{(1)}_{1\,x_1/x_2} + x_2^{m-2} x_3^2 \phi^{(1)}_{2\,x_1/x_2} + \cdots + x_3^m \phi^{(1)}_{m\,x_1/x_2}$$
$$+ x_2^{m-1} x_4 \phi^{(2)}_{1\,x_1/x_2} + x_2^{m-2} x_4^2 \phi^{(2)}_{2\,x_1/x_2} + \cdots + x_4^m \phi^{(2)}_{m\,x_1/x_2}$$
$$+ \Psi_{4m}\,,$$

where

$$x_2^{m-k} \phi^{(j)}_{k\,x_1/x_2} = a_{m-k\,0\,\overbrace{0\ldots0}^{j-1}\,k\,\overbrace{0\ldots0}^{2-j}}\, x_1^{m-k} + a_{m-k-1\,1\,\overbrace{0\ldots0}^{j-1}\,k\,\overbrace{0\ldots0}^{2-j}}\, x_1^{m-k-1} x_2 + \cdots$$
$$+ a_{m-k-h\,\overbrace{0\ldots0}^{j-1}\,k\,\overbrace{0\ldots0}^{2-j}}\, x_1^{m-k-h} x_2^h + \cdots + a_{0\,m-k\,\overbrace{0\ldots0}^{j-1}\,k\,\overbrace{0\ldots0}^{2-j}}\, x_2^{m-k}.$$

Hence, by means of (6) and the methods of the previous section, we have, after writing the results in homogeneous form, with

$$f_{4m} = \prod_{\iota=1}^{m} (x_1 + r_\iota x_2 + r_{3\iota} x_3 + r_{4\iota} x_4) \qquad (a_{m000} = 1),$$

THEOREM 2: *If r_i is a root of $\phi_{0-r}=0$ of multiplicity α_i $(i=1, 2, \ldots, t;$ $\alpha_1+\alpha_2+\ldots+\alpha_t=m)$ of a quaternary form f_{4m}, then the coefficients r_{j+2i} of its linear factors are the roots of the equation*

$$\prod_{i=1}^{t}\left[\frac{\partial^{\alpha_i}\phi^{(j)}_{0-r_i}}{\partial r_i^{\alpha_i}}r_{j+2}^{\alpha_i}-\alpha_i\frac{\partial^{\alpha_i-1}\phi^{(j)}_{1-r_i}}{\partial r_i^{\alpha_i-1}}r_{j+2}^{\alpha_i-1}+\alpha_i(\alpha_i-1)\frac{\partial^{\alpha_i-2}\phi^{(j)}_{2-r_i}}{\partial r_i^{\alpha-2}}r_{j+2}^{\alpha_i-2}\right.$$
$$\left.+\ldots+(-1)^i\,\underline{|\alpha_i}\,\phi^{(j)}_{\alpha_i-r_i}\right]=0 \quad (j=1, 2;\ \phi^{(1)}_{0-r}=\phi^{(2)}_{0-r}=\phi_{0-r}). \qquad (7)$$

§ 3. *Multiple Roots of Functions $\Phi^{(m)}_\xi$, $\Psi^{(m)}_\eta$.*

Let the m roots of the equation

$$f(x)=a_0x^m+a_1x^{m-1}+\ldots+a_m=0$$

be $x_1, x_2, \ldots, x_m$. Represent the totality of the sums of these roots taken two at a time by ξ_i $(i=1, 2, \ldots, (m)=\frac{1}{2}m(m-1))$, and the totality of the products of the roots taken in pairs by $-\eta_j$ $(j=1, 2, \ldots, (m))$. Let the equations having the sets of quantities ξ_i, η_j for roots be respectively

$$\Phi^{(m)}_\xi=\phi_0\xi^{(m)}+\phi_1\xi^{(m)-1}+\ldots+\phi_{(m)}=0,$$
$$\Psi^{(m)}_\eta=\psi_0\eta^{(m)}+\psi_1\eta^{(m)-1}+\ldots+\psi_{(m)}=0.$$

Then, for convenience of statement in this paper, let the following terminology be adopted: Let $x^2-\xi_1x-\eta_1$ be a quadratic factor of $f(x)$. This factor will be said to be of sum-multiplicity α if $f(x)$ has exactly α quadratic factors in which the coefficient of x is the same number, $-\xi_1$. The factor will be said to be of product-multiplicity β if just β of the factors of $f(x)$ have the same absolute term $-\eta_1$. Again, the factor will be called of multiplicity γ if it is repeated as a whole just γ times in $f(x)$. Evidently if the factor is of multiplicity γ, its sum-multiplicity or its product-multiplicity, or both, may be equal to γ or greater than γ.

Let $x^2-\xi_1x-\eta_1$ be a factor of $f(x)$ of multiplicity α. Suppose its sum-multiplicity and product-multiplicity are both α also. Then x_1, x_2 $(x_1+x_2=\xi_1)$ are roots of multiplicity α of $f(x)$.

Let the roots of $\Phi^{(m)}_\xi=0$ be arranged in a triangular array as follows:

$$\begin{array}{ccccc}
x_1+x_2, & x_1+x_3, & x_1+x_4, & x_1+x_5, & \ldots, \\
 & x_2+x_3, & x_2+x_4, & x_2+x_5, & \ldots, \\
 & & x_3+x_4, & x_3+x_5, & \ldots, \\
 & & & x_4+x_5, & \ldots.
\end{array}$$

Then it is evident that ξ_1 is, *formally*, a root of $\Phi^{(m)}_\xi=0$ of multiplicity

$$1+3+5+\ldots+(2\alpha-1)=\alpha^2.$$

Similarly, η_1 is a root of $\Psi_\eta^{(m)}=0$ of multiplicity α^2. More generally, if we assume

$$\left.\begin{array}{l}\xi_1=\xi_2=\cdots\cdots=\xi_{\alpha_1+1}=\cdots\cdots=\xi_{\alpha_1+\alpha_2}=\xi_{\alpha_1+\alpha_2+1}=\cdots\cdots=\xi_{\alpha_1+\alpha_2+\alpha_3}=\cdots\cdots=\xi_{\sum\limits_{i=1}^{k}\alpha_i},\\ \eta_1=\eta_2=\cdots\cdots=\eta_{\alpha_1}\neq\eta_{\alpha_1+1}=\cdots\cdots=\eta_{\alpha_1+\alpha_2}\neq\eta_{\alpha_1+\alpha_2+1}=\cdots\cdots=\eta_{\alpha_1+\alpha_2+\alpha_3}\neq\cdots\cdots=\eta_{\sum\limits_{i=1}^{k}\alpha_i},\end{array}\right\}\quad(8)$$

where $\sum\limits_{i=1}^{k}\alpha_i=\alpha$, it results that ξ_1 is a root of $\Phi_\xi^{(m)}=0$ of multiplicity

$$\alpha_1^2+\alpha_2^2+\cdots\cdots+\alpha_k^2,$$

whereas $\eta_{\alpha_1+\alpha_2+\cdots\cdots+\alpha_h}$ is, *formally*, a root of $\Psi_\eta^{(m)}$ of multiplicity $\alpha_h^2\,(h=1,2,\ldots\ldots,k)$.

A dual result of the same nature is obtained by interchanging the rôles of ξ_i, η_i in this paragraph.

§ 4. *Multiple Quadratic Factors.*

Assume that the general ternary form f_{3m} is decomposable into $\frac{1}{2}m=g$ quadratic factors (m even):

$$\tau_i=n_ix^2+o_ixy+p_iy^2+q_ix+r_iy+s_i\quad(i=1,2,\ldots\ldots,g).$$

Then
$$\prod_{i=1}^{g}(p_iy^2+o_iyx+n_ix^2)=x^m\,\phi_{0\,y/x}.$$

The coefficients of the quadratic forms τ_i and those of f_{3m} itself are connected by the relations

$$\left(\Sigma\frac{\partial}{\partial n_1}q_1+\Sigma\frac{\partial}{\partial o_1}r_1+2\Sigma\frac{\partial}{\partial q_1}s_1\right)a_{m-k-rr}=(k+1)\,a_{m-k-r-1r}\quad(9)$$
$$(k=0,1,2,\ldots\ldots,m-1).$$

Consider now the case where $r^2+o_1r+n_1$ is a factor of multiplicity 1, and of product-multiplicity 1 and sum-multiplicity α of ϕ_{0r}. Then o_1 is a root of multiplicity α of $\Phi_{-o}^{(m)}$, whereas n_1 is a simple root of $\Psi_{-n}^{(m)}$. (We have taken $p_1=p_2=\cdots\cdots=a_{0m}=1$.)

Let
$$\Phi=\Phi_{-o}^{(m)}(a)\equiv I(n_i,o_i,q_i,\cdots\cdots;o)=I,$$
$$\Psi=\Psi_{-n}^{(m)}(a)\equiv J(n_i,o_i,q_i,\cdots\cdots;n)=J.$$

Then
$$I(n_i,o_i,q_i,\cdots\cdots;o_1)\equiv J(n_i,o_i,q_i,\cdots\cdots;n_1)\equiv 0.$$

In $\Phi=I$ let us replace n_i, o_i, q_i by $n_i+\lambda q_i$, $o_i+\lambda r_i$, $q_i+2\lambda s_i$ respectively, and o by $o+\lambda r$. Then Taylor's expansion becomes, by virtue of (9),

$$\sum_{j=0}^{\infty}\frac{\lambda^j}{\underline{|j}}\left(\Sigma\frac{\partial}{\partial n_1}q_1+\Sigma\frac{\partial}{\partial o_1}r_1+2\Sigma\frac{\partial}{\partial q_1}s_1+\frac{\partial}{\partial o}r\right)^jI$$
$$=\sum_{j=0}^{\infty}\frac{\lambda^j}{\underline{|j}}\left(\Sigma\frac{\partial}{\partial a_{m-k-l\,l}}(k+1)\,a_{m-k-l-1\,l}+\frac{\partial}{\partial o}r\right)^j\Phi.\quad(10)$$

When $o = o_1$, $r = r_1$, the first non-vanishing term of the right-hand series is the term where $j = \alpha$. Hence

THEOREM 3: *The α values of r constituting the set of r-coefficients of the α forms τ_i whose binary quadratic parts are factors of sum-multiplicity α of $x^m \phi_{0\,y/x} = 0$, are the roots of the following equation of order α:*

$$r^\alpha \frac{\partial^\alpha \Phi^{(m)}_{-o_1}}{\partial o_1^{\alpha}} + \binom{\alpha}{1} r^{\alpha-1} \frac{\partial^{\alpha-1} \Delta_k \Phi^{(m)}_{-o_1}}{\partial o_1^{\alpha-1}} + \binom{\alpha}{2} r^{\alpha-2} \frac{\partial^{\alpha-2} \Delta_k^2 \Phi^{(m)}_{-o_1}}{\partial o_1^{\alpha-2}} + \dots + \binom{\alpha}{\alpha} \Delta_k^\alpha \Phi^{(m)}_{-o_1} = 0, \tag{11}$$

where

$$\Delta_k = \sum_k \left(a_{0\,m-k-1} \frac{\partial}{\partial a_{1\,m-k-1}} + a_{1\,m-k-2} \frac{\partial}{\partial a_{2\,m-k-2}} + \dots + a_{m-k-1 0} \frac{\partial}{\partial a_{m-k 0}} \right)(k+1).$$

The corresponding values of q are the roots of the α linear equations

$$q_i \frac{\partial \Psi^{(m)}_{-n_i}}{\partial n_i} + \Delta_0 \Psi^{(m)}_{-n_i} = 0 \quad (i = 1, 2, \dots, \alpha). \tag{12}$$

In the functions Φ, Ψ used here the coefficients $a_0, a_1, a_2, \dots, a_m$ of § 3 are replaced by $a_{0m}, a_{1m-1}, \dots, a_{m0}$, respectively.

Next let $r^2 + o_1 r + n_{\alpha_1+\alpha_2+\dots+\alpha_h}$ be a factor of ϕ_{0r} of multiplicity α_h, and of product-multiplicity α_h $(h = 1, 2, \dots, k)$, and let these factors be all of sum-multiplicity $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k$. Then equations (8), § 3, hold $(\xi_1 = -o_1)$, so that $n_{\alpha_1+\alpha_2+\dots+\alpha_h}$ is a root of $\Psi^{(m)}_{-n}$ of multiplicity α_h^2 $(h = 1, 2, \dots, k)$. Also $r^2 + o_1 r + n_{\alpha_1+\alpha_2+\dots+\alpha_h}$, being of multiplicity α_h in ϕ_{0r}, will be of multiplicity $\alpha_h - \varkappa$ in $\phi_{\kappa r}$ $(\varkappa = 0, 1, \dots, \alpha_h - 1)$. Its product-multiplicity will be the same and its sum-multiplicity will be

$$\sum_{h=1}^{k} (\alpha_h - \varkappa) = \alpha - \varkappa k.$$

Then o_1 is a root of $\Phi^{(m-\kappa)}_{-o}$ of multiplicity

$$\mu_\kappa = (\alpha_1 - \varkappa)^2 + (\alpha_2 - \varkappa)^2 + \dots + (\alpha_k - \varkappa)^2,$$

whereas $n_{\alpha_1+\alpha_2+\dots+\alpha_h}$ is a root of $\Psi^{(m-\kappa)}_{-n}$ of multiplicity $(\alpha_h - \varkappa)^2$.

Now we have

$$\Psi_{-n} = \Psi^{(m)}_{-n}(a) = J(n_i, o_i, \dots; n) = J,$$

and $J(n_i, o_i, \dots; n_j) \equiv 0$ $(j = \alpha_1 + \alpha_2 + \dots + \alpha_h)$. In J let n_i, o_i, q_i, n be replaced by $n_i + \lambda q_i$, $o_i + \lambda r_i$, $q_i + 2\lambda s_i$, $n + \lambda q$ respectively, and we get

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\underline{|j}} \left(\Sigma \frac{\partial}{\partial n_1} q_1 + \Sigma \frac{\partial}{\partial o_1} r_1 + 2 \Sigma \frac{\partial}{\partial q_1} s_1 + \frac{\partial}{\partial n} q \right)^j J$$
$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{\underline{|j}} \left(\Sigma \frac{\partial}{\partial a_{m-k-l-1 l}} (k+1) a_{m-k-l-1 l} + \frac{\partial}{\partial n} q \right)^j \Psi^{(m)}_{-n}. \tag{13}$$

The first term of this series which does not vanish identically when $n = n_j$ is the one where $j = \alpha_h^2$. We get a similar result for r by replacing Ψ_{-n} by Φ_{-o}, $\frac{\partial}{\partial n} q$ by $\frac{\partial}{\partial o} r$, and α_h^2 by $\mu_0 = \alpha_1^2 + \alpha_2^2 + \ldots . + \alpha_k^2$. Thus we have

THEOREM 4: *The complete determination of the quadratic factors τ_i of a form f_{3m}, of general multiplicity, involves the solution for the q- and r-coefficients, of the equations*

$$r^{\mu_0} \frac{\partial^{\mu_0} \Phi_{-o_1}^{(m)}}{\partial o_1^{\mu_0}} + \binom{\mu_0}{1} r^{\mu_0 - 1} \frac{\partial^{\mu_0 - 1} \Delta_k \Phi_{-o_1}^{(m)}}{\partial o_1^{\mu_0 - 1}} + \binom{\mu_0}{2} r^{\mu_0 - 2} \frac{\partial^{\mu_0 - 2} \Delta_k^2 \Phi_{-o_1}^{(m)}}{\partial o_1^{\mu_0 - 2}} + \ldots . + \binom{\mu_0}{\mu_0} \Delta_k^{\mu_0} \Phi_{-o_1}^{(m)} = 0, \tag{14}$$

$$q^{\alpha^2 h} \frac{\partial^{\alpha^2 h} \Psi_{-n_j}^{(m)}}{\partial n_j^{\alpha^2 h}} + \binom{\alpha_h^2}{1} q^{\alpha^2 h - 1} \frac{\partial^{\alpha^2 h - 1} \Delta_k \Psi_{-n_j}^{(m)}}{\partial n^{\alpha^2 h - 1}} + \ldots . + \binom{\alpha_h^2}{\alpha_h^2} \Delta_k^{\alpha^2 h} \Psi_{-n_j}^{(m)} = 0 \tag{15}$$

$$(h = 1, 2, \ldots ., k;\ \mu_0 = \alpha_1^2 + \alpha_2^2 + \ldots . + \alpha_k^2).$$

§ 5. *Semi-Resultant and Discriminant Matrices of Forms Representing n-Lines.*

By a semi-resultant we shall mean a necessary and sufficient matrix condition that an m-line and an n-line, each represented by a ternary form, should have a 1-line in common. The name will probably justify itself inasmuch as a semi-resultant of two ternary forms will be seen to be the natural analogue of a binary resultant. It is, however, seminvariantive instead of invariantive.

The author has proved in another paper* that the general m-line is represented by the ternary form

$$f_{3m} = x_2^m \phi_{0\, x_1/x_2} + x_2^{m-1} x_3 \phi_{1\, x_1/x_2} + D^{-1} (-1)^{\frac{1}{2} m(m-1)} \sum_{j=2}^{m} x_3^j \sum_{i=0}^{m-j} \frac{\Delta_1^{m-j-i} \Delta_2^i R_m}{\underline{|m-j-i}\ \underline{|i}} x_1^{m-j-i} x_2^i = 0, \tag{16}$$

where

$$x_2^{m-k} \phi_{k\, x_1/x_2} = a_{k0} x_1^{m-k} + a_{k1} x_1^{m-k-1} x_2 + \ldots . + a_{k\, m-k} x_2^{m-k} \quad (k = 0, 1),$$

R_m is the resultant of $x_2^m \phi_{0\, x_1/x_2}$ and $x_2^{m-1} \phi_{1\, x_1/x_2}$, D is the discriminant of $x_2^m \phi_{0\, x_1/x_2}$, and

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \ldots . + a_{0\, m-1} \frac{\partial}{\partial a_{1\, m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1\, m-1}} + (m-1) a_{0\, m-1} \frac{\partial}{\partial a_{1\, m-2}} + \ldots . + a_{01} \frac{\partial}{\partial a_{10}}.$$

This result is a consequence of Theorem 1, § 1, when $\alpha_i = 1$.

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* (Note added June 10, 1912.) This paper has been published in *Transactions Amer. Mathematical Society*, Vol. XII, No. 3 (1911), p. 373.

It is therefore natural to expect that *any joint invariant of an m-line and an n-line will be expressible entirely in terms of the two binary forms* $x_2^{m-k}\phi_{k\,x_1/x_2}$ *and the analogous forms from the equation of the n-line.* That this is really the case also follows from the results in § 1. That is, we may write the m-line and n-line equations as follows:

$$f_{3m} = x_2^m \phi_{0\,x_1/x_2} + x_2^{m-1} x_3 \phi_{1\,x_1/x_2} + \cdots\cdot,$$
$$g_{3n} = x_2^n \psi_{0\,x_1/x_2} + x_2^{n-1} x_3 \psi_{1\,x_1/x_2} + \cdots\cdot \quad (n \leqq m).$$

Then we have (Theorem 1)

$$f_{3m} = \prod_{i=1}^{m} \left(x_1 + r_i x_2 - \phi_{1-r_i} \Big/ \frac{\partial \phi_{0-r_i}}{\partial r_i} \right),$$
$$g_{3m} = \prod_{j=1}^{n} \left(x_1 + r_j x_2 - \psi_{1-r_j} \Big/ \frac{\partial \psi_{0-r_j}}{\partial r_j} \right).$$

It now follows without difficulty that the condition for a common 1-line is equivalent to the condition that the following three binary equations have a common root:

$$\left.\begin{aligned} &\phi_{0x} = 0, \qquad \psi_{0x} = 0, \\ &X_x^{m+n-2} = \phi_{1x}\frac{\partial \psi_{0x}}{\partial x} - \psi_{1x}\frac{\partial \phi_{0x}}{\partial x} = 0. \end{aligned}\right\} \quad (17)$$

This condition can be expressed by means of a matrix, by using Stuyvaert's generalization* of the dialytic eliminant. Stuyvaert has shown that a necessary and sufficient condition that three polynomials in x of respective orders λ, μ, ν, $\nu \leqq \lambda + \mu - 1$, may have a common root, is that a certain matrix of $\lambda + \mu$ rows and $\lambda + \mu + 1$ columns should be of rank† $\lambda + \mu - 1$. We may express the condition here desired in the form of such a matrix with

$$\lambda = m - 1, \quad \mu = n, \quad \nu = m + n - 2.$$

* Stuyvaert: *Cinq études de géométrie analytique* (1908), p. 60.

† Conditions more desirable than $J_2 = J_3 = 0$ in § 1, part II, of my former paper can be obtained by constructing from the elements of J_2, J_3, a Stuyvaert matrix, $\lambda = 2$, $\mu = 2$, $\nu = 3$, *e. g.*,

$$M = \left\| \begin{matrix} a & 0 & a' & 0 & a_0 \\ b & a & b' & a' & a_1 \\ c & b & c' & b' & a_2 \\ 0 & c & 0 & c' & a_3 \end{matrix} \right\|,$$

where the sets a, b, c, and a', b', c' are respectively the non-zero elements of the third rows of J_2 and J_3. The necessary conditions $J_2 = J_3 = 0$ are also sufficient if $a^3_x = 0$ has but one real root. But the rank of M being 3 is both a necessary and sufficient condition, free from any assumption.

Thus, suppose $m=3$, $n=2$. Then

$$\phi_{0x}=a_{00}\,x^3+a_{01}\,x^2+a_{02}\,x+a_{03},\quad \psi_{0x}=b_{00}\,x^2+b_{01}\,x+b_{02},$$
$$X_x^3=X_0\,x^3+X_1\,x^2+X_2\,x+X_3,$$

where

$$\begin{aligned}X_0&=2\,a_{10}\,b_{00}-3\,a_{00}\,b_{10},\\ X_1&=a_{10}\,b_{01}-3\,a_{00}\,b_{11}+2\,a_{11}\,b_{00}-2\,a_{01}\,b_{10},\\ X_2&=a_{11}\,b_{01}-2\,a_{01}\,b_{11}+2\,a_{12}\,b_{00}-a_{02}\,b_{10},\\ X_3&=a_{12}\,b_{01}-a_{02}\,b_{11}.\end{aligned}$$

Then the semi-resultant of f_{33} and g_{32} is*

$$\rho_{32}\equiv\left\|\begin{matrix}a_{01}\,b_{00}-a_{00}\,b_{01} & a_{02}\,b_{00}-a_{00}\,b_{02} & a_{03}\,b_{00} & 0\\ 0 & a_{01}\,b_{00}-a_{00}\,b_{01} & a_{02}\,b_{00}-a_{00}\,b_{02} & a_{03}\,b_{00}\\ b_{00} & b_{01} & b_{02} & 0\\ 0 & b_{00} & b_{01} & b_{02}\\ X_0 & X_1 & X_2 & X_3\end{matrix}\right\|$$

THEOREM 5: *A necessary and sufficient condition*† *that* f_{3m}, g_{3n} *may have a common linear factor is that* ρ_{mn} *should be of rank* $m+n-2$.

It is worthy of notice that the coefficients of ϕ_{1x}, ψ_{1x} occur only in the X_i, so that two fourth-order determinants of ρ_{32} are linear in these coefficients. Therefore two of the latter are expressible rationally in terms of the remaining coefficients in the matrix.

THEOREM 6: *In general, if an m-line and an n-line have a* 1*-line in common, there will be* $2\,m-1$ *independent coefficients in* f_{3m} *and* $2\,n-1$ *in* g_{3n}. *Moreover, the remaining* $\frac{1}{2}(m^2-m+2)$ *coefficients of the first form and* $\frac{1}{2}(n^2-n+2)$ *of the second are all rationally expressible in terms of the* $2\,(m+n-1)$ *independent ones.*

The condition that a given m-line $f_{3m}=0$ may have a double 1-line can be expressed as a Stuyvaert matrix condition that the three binary equations

$$\phi_{0x}=0,\quad \phi'_{0x}=0,\quad \nabla_x^{2m-4}=\phi_{1x}'^2-2\,\phi''_{0x}\,\phi_{2x}=0\ \ddagger \tag{18}$$

* The negative condition

$$\left\|\begin{matrix}a_{01}\,b_{00}-a_{00}\,b_{01} & a_{02}\,b_{00}-a_{00}\,b_{02} & a_{03}\,b_{00}\\ b_{00} & b_{01} & b_{02}\end{matrix}\right\|\neq 0$$

is to be understood.

† As Stuyvaert points out, a certain negative condition concerning non-vanishing first minors of a definite determinant of order $\mu+\nu$ of the matrix is to be understood (see ρ_{32} above).

‡ Theorem 1 with $a_1=2$, $a_2=1$, $m=3$.

may have a common root. When $m = 3$, this matrix is

$$\delta_3 \equiv \left\| \begin{array}{ccc} 3\,a_{00} & 2\,a_{01} & a_{02} \\ 6\,a_{00}\,a_{02} - 2\,a_{01}^2 & 9\,a_{00}\,a_{03} - a_{02}a_{01} & 0 \\ 0 & 6\,a_{00}\,a_{02} - 2\,a_{01}^2 & 9\,a_{00}\,a_{03} - a_{02}\,a_{01} \\ 4\,a_{10}^2 - 12\,a_{00}\,a_{20} & 4\,a_{10}\,a_{11} - 12\,a_{00}\,a_{21} - 4\,a_{01}\,a_{20} & a_{11}^2 - 4\,a_{01}\,a_{21} \end{array} \right\|$$

THEOREM 7: *A necessary and sufficient condition that the m-line* $f_{3m} = 0$ *may have a double* 1*-line is that the matrix* δ_m *should be of rank* $2\,m - 4$. *The form* f_{3m} *then has but* $2\,m - 2$ *independent coefficients and the remainder are all rationally expressible in terms of those of* ϕ_{0x}, ϕ_{1x}.

Inasmuch as δ_m gives the condition for a repeated linear factor of f_{3m}, it might be called the *ultra-discriminant.* Generalizations giving conditions for a common 1-line for more than two m-lines, and corresponding results for p-ary forms $p > 3$, can be made readily.

Moreover, the method given may be extended in the direction of finding a necessary and sufficient condition for a 1-line of multiplicity greater than 2 of an m-line. Thus from Theorem 1 with $\alpha_1 = 3$ the condition for a triple 1-line is expressible as a Stuyvaert matrix (generalized) condition that the following five binary equations should have a common root:

$$\left.\begin{array}{l} \phi_{0x} = 0, \quad \phi'_{0x} = 0, \quad \phi''_{0x} = 0, \\ \rho_x^{2m-6} = \phi_{1x}^{\prime\prime 2} - 2\,\phi'''_{0x}\,\phi'_{2x} = 0, \\ \sigma_x^{2m-6} = 2\,\phi_{2x}^{\prime 2} - 3\,\phi''_{1x}\,\phi_{3x} = 0. \end{array}\right\} \quad (19)$$

These are derived from the Hessian of (4) with $\alpha_1 = 3$.

§ 6. *Resolution into Rational Partial Fractions.*

The results developed in sections 1, 3, 4 give us general formulas for the resolution of an ordinary rational proper algebraical fraction into the various standard types of partial fractions. For evidently the problem of resolving $x_2^{m-1}\,\phi_{1\,x_1/x_2} / x_2^m\,\phi_{0\,x_1/x_2}$ into partial fractions with real (linear and quadratic) denominators is identical, in part, with that of factoring a factorable form

$$x_2^m\,\phi_{0\,x_1/x_2} + x_2^{m-1}\,\phi_{1\,x_1/x_2} + \cdots .$$

into its linear and quadratic real ternary factors (see § 3). We can consider

the general partial fraction theorem, therefore, as one interpretation of theorems 3, 4. A sufficiently general statement of it is embodied in the following:

THEOREM 8: *Let* $x_2^m \phi_{0\, x_1/x_2} = \prod_{i=1}^{h} (x_1 + r_i x_2) \prod_{j=1}^{l} (x_1^2 + \xi_j x_1 x_2 + \eta_j x_2^2)$, *where* $x_1 + r_i x_2$ *is a single (not multiple) factor and* $x_1^2 + \xi_j x_1 x_2 + \eta_j x_2^2$ *is of multiplicity* 1, *sum-multiplicity* α_j *and product-multiplicity* 1. *Then we have the resolution*

$$\frac{x_2^{m-1} \phi_{1\, x_1/x_2}}{x_2^m \phi_{0\, x_1/x_2}} = \sum_{i=1}^{h} \frac{\nu_i}{x_1 + r_i x_2} + \sum_{j=1}^{l} \frac{\lambda_j x_1 + \mu_j x_2}{x_1^2 + \xi_j x_1 x_2 + \eta_j x_2^2},$$

where

$$\nu_i = \frac{\Delta_k \phi_{0-r_i}}{\Delta \phi_{0-r_i}}, \quad \mu_j = -\Delta_k \Psi_{-\eta_j}^{(m)} / \Delta \Psi_{-\eta_j}^{(m)},$$

and the numbers λ_j *corresponding to the* α_j *factors of sum-multiplicity* α_j *are the* α_j *roots of*

$$\lambda^{\alpha_j} \frac{\partial^{\alpha_j} \Phi_{-\xi_j}^{(m)}}{\partial \xi_j^{\alpha_j}} + \binom{\alpha_j}{1} \lambda^{\alpha_j - 1} \frac{\partial^{\alpha_j - 1} \Delta_k \Phi_{-\xi_j}^{(m)}}{\partial \xi_j^{\alpha_j - 1}} + \binom{\alpha_j}{2} \lambda^{\alpha_j - 2} \frac{\partial^{\alpha_j - 2} \Delta_k^2 \Phi_{-\xi_j}^{(m)}}{\partial \xi_j^{\alpha_j - 2}} + \dots + \binom{\alpha_j}{\alpha_j} \Delta_k^{\alpha_j} \Phi_{-\xi_j}^{(m)} = 0,$$

where

$$\Delta = \frac{\partial}{\partial r}, \quad \Delta_k = \sum_k \left(a_{0\, m-k-1} \frac{\partial}{\partial a_{1\, m-k-1}} + a_{1\, m-k-2} \frac{\partial}{\partial a_{2\, m-k-2}} + \dots + a_{m-k-1\, 0} \frac{\partial}{\partial a_{m-k\, 0}} \right) (k+1).$$

These methods may be extended to the case of rational p-ary fractions $p > 2$.

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